LINK CONCORDANCE, BOUNDARY LINK CONCORDANCE AND ETA-INVARIANTS

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Abstract. We study the eta-invariants of links and show that in many cases they form link concordance invariants, in particular that many eta-invariants vanish for slice links. This result contains and generalizes previous invariants by Smolinsky and Cha–Ko. We give a formula for the eta-invariant for boundary links. In several intersting cases this allows us to show that a given link is not slice. We show that even more eta-invariants have to vanish for boundary slice links. We give an example of a boundary link L that is not boundary slice but where all the known link concordance invariants computed so far are zero.

1. Introduction

An m-link of dimension n is an embedded oriented smooth submanifold of S^{n+2} that is homeomorphic to m ordered copies of S^n. A link concordance between two given links in S^{n+2} is a properly embedded oriented submanifold in $S^{n+2} \times [0,1]$ that is homeomorpic to m copies of $S^n \times [0,1]$ and intersects $S^{n+2} \times 0$ and $S^{n+2} \times 1$ at the given links. We say a link is slice if it is concordant to the trivial link. Equivalently a link is slice if it bounds m disjoint smooth disks in D^{n+3}.

Denote by $C(n,m)$ the set of concordance classes of m-links of dimension n. The set $C(n,1)$ is just the set of knot concordance classes, it has a well-defined group structure given by connected sum along arcs. Connected sum of links does not give a well-defined group structure on $C(n,m)$ since there's no canonical choice of arcs (cf. proposition 5.1).

It is very difficult to determine $C(n,m)$, a common approach is to study links with some extra structure. A boundary link is an m-link which has m disjoint Seifert manifolds, i.e. there exist m disjoint oriented $(n+1)$-submanifolds $V_1, \dots, V_m \subset S^{n+2}$ such that $\partial(V_i) = L_i, i = 1, \dots, m$. A boundary link concordance between two given boundary links in S^{n+2} is a link concordance which bounds m disjoint $(n+2)$-manifolds in $S^{n+2} \times [0,1]$. We say L is boundary slice if it is boundary concordant to the unlink. Denote by $B(n,m)$ the set of boundary concordance classes of m-boundary links of dimension n.

A pair (L,V) consisting of a boundary link and a Seifert manifold is called boundary link pair. There's an obvious notion of concordance for boundary link pairs. Denote by $C_n(B_m)$ the set of concordance classes of boundary link pairs. Assume $n > 1$ and

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let (L_1, V_1) and (L_2, V_2) be representatives of elements in $C_n(B_m)$. We can assume that V_1, V_2 are simply connected, and then $S^{n+2} \setminus (V_1 \cup V_2)$ is simply connected, in particular up to homotopy there's only one arc connecting L_1, L_2 in the complement of $V_1 \cup V_2$, therefore the boundary connected sum $(L_1 \# L_2, V_1 \# V_2) \in C_n(B_m)$ is well-defined and turns $C_n(B_m)$ into a group.

Let F_m be the free group on the generators $t_1, \dots, t_m$. An F_m-link is a pair (L, φ) where L is a link in S^{n+2} and $\varphi : \pi_1(S^{2n+2} \setminus L) \to F_m$ is an epimorphism sending an i^{th} meridian to t_i. A pair (N, Φ) is an F_m-concordance between (L_0, φ_0) and (L_1, φ_1) if M is a link concordance between the links L_0 and L_1 and $\Phi : \pi_1(S^{n+2} \times [0,1] \setminus N) \to F_m$ is a map extending φ_0 and φ_1 up to inner automorphisms (cf. [CS80]). Denote by $C_n(F_m)$ the set of F_m-concordance classes of F_m-links. If $n > 1$ then any element in $C_n(F_m)$ has a representative (L, φ) such that φ is an isomorphism, this φ defines canonical meridians for L, which can be used to defined a well-defined group structure on $C_n(F_m)$.

By the transversality argument, such an epimorphism φ gives a Seifert surface V_φ. Conversely, the existence of a Seifert surface V for L produces such an epimorphism φ_V by the Thom-Pontryagin construction. We'll freely go back and forth between isotopy classes of boundary link pairs (L, V) and F_m-links (L, φ). Similarly there's an equivalence between the respective concordances, in particular $C_n(B_m) \cong C_n(F_m)$, which is a group isomorphism for $n > 1$.

We say that $\varphi : \pi_1(S^{n+2} \setminus L) \to F_m$ is a splitting map if it sends meridians to generators. There's in general not a unique splitting map. Denote by CA_m the group of automorphisms of F_m which send t_i to a conjugate of t_i for each $i = 1, \dots, m$.

Lemma 1.1. *If $\varphi : \pi_1(S^{n+2} \setminus L) \to F_m$ is a splitting map, then for any $\phi \in CA_m$ the map $\phi \circ \varphi$ is a splitting map as well, and in fact all splitting maps are of the form $\phi \circ \varphi$ for some $\phi \in CA_m$.*

This means that we have an action of CA_m on $C_n(F_m)$. The inner automorphisms of F_m are elements in CA_m and act trivially on $C_n(F_m)$. We therefore define A_m to be the quotient group of CA_m by the inner automorphisms of F_m. We get an action of A_m on $C_n(F_m)$. Denote by $\phi_{ij} : F_m \to F_m$ the map which sends t_i to $t_j t_i t_j^{-1}$ and t_k to t_k for $k \neq i$. We quote the following proposition (cf. [K84], [K87]).

Proposition 1.2. *CA_m (and in particular A_m) is generated by ϕ_{ij} for $i, j = 1, \dots, m$ and $i \neq j$. Furthermore the groups A_1, A_2 are trivial.*

Under the isomorphism $C_n(F_m) \cong C_n(B_m)$ the group A_m also acts on $C_n(B_m)$, the action of ϕ_{ij} on a Seifert surface has been described explicitely by Ko [K87]. Ko [K87] furthermore showed that A_m acts non-trivially on $C_n(B_m)$ and hence acts non-trivially on $C_n(F_m)$.

Theorem 1.3. [CS80]

$$B(n, m) \cong C_n(F_m)/A_m \cong C_n(B_m)/A_m$$

Cappell and Shaneson showed that $C_{2k}(F_m)=0$, i.e. all even dimensional boundary links are boundary slice. It is not known whether all even dimensional (boundary) links are slice. We'll restrict ourselves from now on to odd-dimensional links.

For $\epsilon=\pm 1$ we call $A=(A_{ij})_{i,j=1,\dots,m}$ an ϵ-boundary link Seifert matrix of size $(g_1,\dots,g_m)$ if A is a matrix with entries A_{ij} which are $(2g_i\times 2g_j)$–matrices over $\mathbb{Z}$ such that $A_{ij}=-\epsilon A_{ji}^t$ for $i\neq j$ and $\det(A_{ii}+\epsilon A_{ii}^t)=1$ (cf. [L77], [K87]). We say that A_{ij} is metabolic if there exists a block diagonal matrix $P=\operatorname{diag}(P_1,\dots,P_m)$ such that each $P_iA_{ij}P_j^t$ is of the form

$$\begin{pmatrix} 0 & C \\ D & E\end{pmatrix}$$

where 0 is a $g_i\times g_j$-matrix. This generates in a natural way an equivalence class of matrices, the set of equivalence classes is denoted by $G(m,\epsilon)$.

If $n=2q-1$ then picking a basis for the torsion free parts of $H_q(V)=H_q(V_1)\oplus\dots\oplus H_q(V_m)$ we can associate to a boundary link pair (L,V) the matrix representing the Seifert pairing

$$\begin{array}{rcl} H_q(V)\times H_q(V) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \operatorname{lk}(a,b_+)\end{array}$$

Theorem 1.4. [K85]

(1) *Every Seifert matrix is the Seifert matrix of a boundary link pair,*
(2) *for* $q\geq 3$
$$C_{2q-1}(B_m)\cong G(m,(-1)^q)$$
(3) $C_3(B_m)$ *is isomorphic to a subgroup of* $G(m,1)$ *of index* 2^m.

The A_m action on $C_{2q-1}(B_m)$ translates to an action of A_m on $G(m,(-1)^q)$ which was explicitely computed by Ko [K87]. Summarizing we get for $q\geq 3$ that

$$B(2q-1,m)\cong C_{2q-1}(B_m)/A_m\cong G(m,(-1)^q)/A_m$$

Levine [L69b] showed that $G(1,\epsilon)\cong\mathbb{Z}^{\oplus\infty}\oplus\mathbb{Z}_2^{\oplus\infty}\oplus\mathbb{Z}_4^{\oplus\infty}$ (cf. also [S77]). Recently Sheiham [S02] showed that for $m>1$, $G(m,\epsilon)\cong\mathbb{Z}^{\oplus\infty}\oplus\mathbb{Z}_2^{\oplus\infty}\oplus\mathbb{Z}_4^{\oplus\infty}\oplus\mathbb{Z}_8^{\oplus\infty}$, furthermore Sheiham defined full invariants for $G(m,\epsilon)$.

A lot of effort has been put into the study of the forgetful map

$$B(n,m)\to C(n,m)$$

Cochran and Orr [CO90], [CO93], Gilmer and Livingston [GL92] and Levine [L94] showed that this map is not surjective, i.e. there exist links which are not concordant to boundary links. It is an open question whether the kernel is trivial, i.e. whether any knot that is slice is also boundary slice. It would be very difficult to find counter-examples in dimension one, since one can easily see that any ribbon (boundary) link is boundary slice.

Given a closed smooth odd dimensional manifold M and a unitary representation $\alpha:\pi_1(M)\to U(k)$, Atiyah–Patodi–Singer [APS75] defined an invariant $\eta_\alpha(M)\in\mathbf{R}$,

called the eta–invariant, which can be computed in terms of signatures of bounding manifolds, if these exist. For a group G a pair (M,φ) is called a G-manifold if M is a smooth odd-dimensional manifold and $\varphi:\pi_1(M)\to G$ a homomorphism. Define $\rho(M,\varphi):R_k(G)\to\mathbf{R}$ via $\rho(M,\varphi)(\alpha):=\eta_{\alpha\circ\varphi}(M)$. Two G-manifolds $(M_j,\alpha_j), j=1,2$ are called homology G-bordant if there exists a G-manifold (N,β) such that $\partial(N)=M_1\cup -M_2, H_*(N,M_j)=0$ for $j=1,2$ and, up to inner automorphisms of G, $\beta|\pi_1(M_j)=\alpha_j$.

Theorem 1.5. [L94, p. 95] *If $(M_i,\alpha_i), i=1,2$ are homology G-bordant manifolds, then $\rho(M_1,\varphi_1)(\alpha)=\rho(M_2,\varphi_2)(\alpha)$ for all $\alpha:G\to U(k)$ that factor through a p–group.*

We'll study the ρ–invariant for M_L, the result of zero framed surgery along $L\subset S^{2q+1}$. For G a group define the lower central series inductively by $G_0:=G, G_i:=[G,G_{i-1}]$. For the remainder of the introduction we'll denote the free group on m generators by F. For an m-component link $L\subset S^{2q+1}$ we have in many cases (e.g. if $q>1$) an isomorphism $\pi_1(S^{2q+1}\setminus L)/\pi_1(S^{2q+1}\setminus L)_i\to F/F_i$. A choice of isomorphism is called an F/F_i–structure. Two links L_1,L_2 with F/F_i–structures that are concordant also have concordant F/F_i–structures, and M_{L_1} and M_{L_2} have homology F/F_i–bordant F/F_i–structures. Applying the above theorem gives a link concordance obstruction theorem. The theory becomes even easier if we want to find sliceness obstructions since any slice knot has an F/F_i–structure for all i and since any representation factoring through a p–group factors through F/F_i for some i.

Theorem 1.6. *Let $L\subset S^{2q+1}$ be a slice link, if $\alpha:\pi_1(M_L)\to U(k)$ factors through a p–group, then $\eta_\alpha(M_L)=0$.*

Define $PD(k)\subset U(k)$ to be the subgroup generated by permutation matrices and diagonal matrices. For a prime p define $PD_p(k)\subset PD(k)$ to be the subgroup of matrices where all eigenvalues are roots of unity of order a power of p.

Theorem 1.7. *Let $L\subset S^{2q+1}$ be a slice link with meridians $\mu_1,\dots,\mu_m$. Let p be a prime number and let $U_1,\dots,U_m\in PD_p(K)$. Then there exists a unique representation $\beta:\pi_1(M_L)\to U(k)$ with $\beta(\mu_j)=U_j$. Furthermore $\eta_\beta(M_L)=0$.*

This gives the best possible sliceness obstruction theorem that can be based on Levine's theorem. These obstructions combine, simplify and generalize sliceness obstructions defined by Smolinsky [S89], [S89b] and Cha and Ko [CK99].

For an F–link (L,φ) the ρ-invariant can be explicitely computed in terms of its Seifert matrix. In the case $n=4q+3$ the following holds, the case $n=4q+1$ being only marginally more complicated (cf. theorem 4.5).

Theorem 1.8. *Let $(L\subset S^{4q+3},\varphi)$ be an F_m-link, $A=(A_{ij})_{i,j=1,\dots,m}$ a Seifert matrix, $\alpha:F_m\to U(k)$ a representation. Let $U_i:=\alpha(t_i)$, then $\rho(M_L,\varphi)(\alpha)=sign(M(A,\alpha))$*

where $M(A,\alpha)$ equals

$$\begin{pmatrix} A_{11}\otimes(id-U_1^{-1})+A_{11}^t\otimes(id-U_1) & A_{12}\otimes(id-U_1)(id-U_2^{-1}) & \dots \\ A_{21}\otimes(id-U_2)(id-U_1^{-1}) & A_{22}\otimes(id-U_2^{-1})+A_{22}^t\otimes(id-U_2) & \\ \vdots & & \ddots \end{pmatrix}$$

This formula makes it possible to compute enough ρ–invariants to show that several interesting boundary links are neither boundary link slice nor slice. Note that if L is boundary link slice then $\rho(M_L,\varphi)(\alpha)$ for all representations α with $\det(M(A,\alpha))=0$, i.e. not only for representations that factor through a p–group. Levine announced a proof that this result also holds in the case that L is slice.

The structure of this paper is as follows. In section 2 we'll give a more detailed exposition of the eta-invariant and the rho-invariant. In particular we'll cite a criterion of Levine's when homology G–bordant manifolds have identical eta-invariants. These results will be applied in section 3 to link concordance questions and in section 4 to boundary link concordance questions. We furthermore define a useful signature function for boundary links. We apply our invariants to several interesting cases in section 5. We conclude the paper with two sections containing a formula relating eta-invariants of finite covers and the computation of the ρ-invariant for boundary links.

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2. The eta invariant as cobordism invariant

Let M^{2q+1} be a closed odd-dimensional smooth manifold and $\alpha:\pi_1(M)\to U(k)$ a unitary representation. Atiyah, Patodi, Singer [APS75] associated to (M,α) a number $\eta_\alpha(M)$ called the (reduced) eta-invariant of (M,α). For more details cf. section 6.

For a hermitian matrix or form A (i.e. $\bar{A}^t=A$) we define

$$\mathrm{sign}(A) := \#\text{ positive eigenvalues of } A - \#\text{ negative eigenvalues of } A$$

and for a skew-hermitian matrix A (i.e. $\bar{A}^t=-A$) we define $\mathrm{sign}(A):=\mathrm{sign}(iA)$.

The main theorem to compute the eta-invariant is the following (cf. [APS75]).

Theorem 2.1. *(Atiyah-Patodi-Singer index theorem) Let (M^{2q+1},α) as above. If there exists $(W^{2q+2},\beta:\pi_1(W)\to U(k))$ with $\partial(W^{2q+2},\beta)=r(M^{2q+1},\alpha)$ for some $r\in\mathbb{N}$, then*

$$\eta_\alpha(M)=\frac{1}{r}(sign_\beta(W)-k\,sign(W))$$

Let G be a group, then a G-manifold is a pair (M,φ) where M is a compact oriented manifold with components $\{M_i\}$ and φ is a collection of homomorphisms $\varphi_i:\pi_1(M_i)\to G$ where each φ_i is defined up to inner automorphism. Let $R_k(G):=$

$\{\alpha | G \to U(k)\}$. For an odd-dimensional G-manifold (M, φ) define

$$\begin{array}{rcl} \rho(M,\varphi) : R_k(G) & \to & \mathbf{R} \\ \alpha & \mapsto & \eta_{\alpha\circ\varphi}(M) \end{array}$$

We call two odd-dimensional G-manifolds $(M_j, \alpha_j), j = 1, 2$, homology G-bordant if there exists a G-manifold (N, β) such that $\partial(N) = M_1 \cup -M_2, H_*(N, M_j) = 0$ for $j = 1, 2$ and, up to inner automorphisms of G, $\beta|\pi_1(M_j) = \alpha_j$. We want to relate the ρ-function for homology G-bordant manifolds.

Let

$$P_k(G) = \{\alpha \in R_k(G) | \alpha \text{ factors through a group of prime power order}\}$$

Theorem 2.2. [L94, p. 95] *If $(M_i, \alpha_i), i = 1, 2$ are homology G-bordant manifolds, then*

$$\rho(M_1, \varphi_1)(\alpha) = \rho(M_2, \varphi_2)(\alpha) \text{ for all } \alpha \in P_k(G)$$

3. Eta invariants as link concordance invariants

Let $L \subset S^{2q+1}$ be a link. We'll study the eta-invariants associated to the closed manifold M_L, the result of zero-framed surgery along $L \subset S^{2q+1}$. We first compute the eta invariants of the trivial link.

Lemma 3.1. *Let M_O be the zero-framed surgery on the trivial link L. Then for any $\alpha : \pi_1(M_O) \to U(k)$ we get $\eta_\alpha(M_O) = 0$.*

Proof. Let $\alpha : \pi_1(M_O) \to U(k)$ be a representation. Let $D_1, \dots, D_m$ be the push-in off the disks in S^{2q+1} bounding $L_1, \dots, L_m$ and let $W := D^{2q+2} \setminus (N(D_1) \cup \dots \cup N(D_m))$. Note that $\pi_1(S^{2q+1} \setminus L) \cong \pi_1(W) \cong F$, in particular we can use W to compute $\eta_\alpha(M_O)$. But W is homotopy equivalent to the wedge of m circles, in particular $H_{q+1}(W) = H^\alpha_{q+1}(W, \mathbb{C}^k) = 0$, hence the untwisted and twisted signatures vanish, hence $\eta_\alpha(M_O) = 0$ by theorem 2.1. □

3.1. **Abelian eta invariants.** Recall that any oriented link L with m components has a canonical map $\epsilon_L : \pi_1(M_L) \to H_1(M_L) = \mathbb{Z}^m$. Furthermore if L_1, L_2 are link concordant, then (M_{L_1}, ϵ) and (M_{L_1}, ϵ) are canonically homology $\mathbb{Z}^m$-bordant.

The following is now immediate from theorem 2.2.

Proposition 3.2. *Let L_1, L_2 be concordant links, then*

$$\rho(M_{L_1}, \epsilon)(\alpha) = \rho(M_{L_2}, \epsilon)(\alpha) \text{ for all } \alpha \in P_k(\mathbb{Z}^m)$$

The following corollary contains basically the statement of Smolinsky's main theorem in [S89b]. It follows immediately from the proposition and lemma 3.1.

Corollary 3.3. *Let L be a slice link, $\alpha \in P_1(\mathbb{Z}^m)$, then $\eta_\alpha(M_L) = 0$.*

Remark. Levine [L94] shows that that there are links whose eta-invariants vanish for all $\alpha \in P_1(\mathbb{Z}^m)$ but where a close study of $\rho(M, \epsilon) : R_1(\mathbb{Z}^2) \to \mathbf{R}$ still shows that the links are not slice.

We quickly recall a result from high-dimensional knot theory. Combining results of Matumuto [M77] and Levine [L69], [L69b] we get the following theorem.

Theorem 3.4. *If $q > 1$, then a knot $K \subset S^{2q+1}$ represents a torsion element in $C(2q-1, 1)$ if and only if $\eta_\alpha(M_K) = 0$ for all $\alpha \in P_1(\mathbb{Z})$.*

In section 5 we show that one–dimensional eta–invariants are not enough to detect non–torsion elements in $C_{2q-1}(B_m)$ for $m > 1$ and $q > 1$.

3.2. **Non–abelian eta invariants.** For G a group define the lower central series inductively by $G_0 := G$ and $G_i := [G, G_{i-1}], i > 0$. Milnor [M57] showed that for a link L

$$\pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)_k \cong \langle x_1, \dots, x_m | [x_i, w_i], \langle x_1, \dots, x_m \rangle_k \rangle$$

where x_i are representatives for the meridians, w_i for the longitudes and $\langle x_1, \dots, x_m \rangle_k$ denotes the k^{th} term in the lower central series of the free group generated by $x_1, \dots, x_m$.

To avoid confusion we'll henceforth denote the free group on m generators $t_1, \dots, t_m$ by F. Let $F \to \pi_1(S^{2q+1} \setminus L) =: \pi$ be a map t_i to a meridian of the i^{th} component of L. Levine [L94] shows that this induces isomorphisms $F/F_i \xrightarrow{\cong} \pi/\pi_i$ for all i if $q > 1$. If $q = 1$, then we say that L has zero $\bar{\mu}$–invariant of level i if this induces an isomorphism $F/F_i \xrightarrow{\cong} \pi/\pi_i$. By Milnor's result on $\pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)_k$ a knot has zero $\bar{\mu}$–invariant of level i if and only if for longitudes $\lambda_1, \dots, \lambda_m$, $\{\lambda_j\} \in \pi_1(S^{2q+1} \setminus L)_i$. Examples for 1-dimensional links with zero $\bar{\mu}$–invariants are boundary links.

We say $\varphi : \pi_1(S^{n+2} \setminus L) \to F/F_i$ is an F/F_i–structure if a meridian of the j^{th} component gets sent to t_j. Note that it follows from Stalling's theorem [S65] that conjugates of generators for F/F_i are also generators of F/F_i.

The case $i = 1$ is of course uninteresting since $F/F_i = \mathbb{Z}^m$. If $i > 1$ then L has in general no canonical F/F_i–structure.

Lemma 3.5. [L94, p. 101] *If φ_1 and φ_2 are F/F_i–structures for the same link, then $\varphi_1 = \psi \circ \varphi_2$ for an automorphism of F/F_i that sends t_j to a conjugate of $t_j, j = 1, \dots, m$*

We call such an automorphism a special automorphism of F/F_i. A link L equipped with an F/F_i–structure is called F/F_i–link. Let $(L_1, \varphi_1), (L_2, \varphi_2)$ be two F/F_i–links, we say they are F/F_i–concordant if there exists a link concordance C and a map $\varphi : \pi_1(S^{2q+1} \times [0, 1] \setminus C) \to F/F_i$ which restricts to φ_1 and φ_2 up to inner automorphism.

The following proposition is well–known.

Proposition 3.6. (1) *If L_1 is an F/F_i–link and L_2 is link concordant to L_1, then there exists an F/F_i–structure on L_2 such that L_1 and L_2 are F/F_i–concordant.*

(2) *If L_1, L_2 are link concordant and L_1 has zero $\bar{\mu}$–invariants of level j, then L_2 also has zero $\bar{\mu}$–invariants of level j.*
(3) *A one-dimensional slice link has zero $\bar{\mu}$–invariant for all levels.*

Proof. Let $C \subset S^{2q+1} \times [0,1]$ be a link concordance between L_1 and L_2.

(1) Consider
$$\pi^j := \pi_1(S^{2q+1} \setminus L_j) \to \pi_1(S^{2q+1} \times [0,1] \setminus C) =: \pi_C$$
These maps are normally surjective and hence define isomorphisms $\pi_C/\pi_{C,i} \cong \pi^j/\pi^j_i \cong F/F_i$ by Stalling's theorem [S65]. The statement now follows easily (cf. [L94, p. 102] for details).
(2) This follows immediately from the definition and $F/F_i \cong \pi^1/\pi^1_i \cong \pi_C/\pi_{C,i} \cong \pi^2/\pi^2_i$.
(3) This follows immediately from (2) since a slice link is concordant to the unlink which has obviously zero $\bar{\mu}$-invariant for all levels.

□

It is clear that in the case $q > 1$ the map $\pi_1(S^{2q+1} \setminus L) \to \pi_1(M_L)$ is an isomorphism, hence
$$\pi_1(M_L)/\pi_1(M_L)_i \cong \pi_1(S^{2q+1} \setminus L)/\pi_1(S^{2q+1} \setminus L)_i$$
If $q = 1$ the kernel $\pi_1(S^3 \setminus L) \to \pi_1(M_L)$ is generated by the longitudes. In particular if L has zero $\bar{\mu}$-invariants of level i, then
$$\pi_1(M_L)/\pi_1(M_L)_i \cong \pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)_i$$
In both cases an F/F_i–structure on L gives an F/F_i–structure on M_L.

Proposition 3.7. [L94, p. 102] *If $(L_1, \varphi_1), (L_2, \varphi_2)$ are F/F_i–concordant F/F_i–links, then (M_{L_1}, φ_1) and (M_{L_2}, φ_2) are homology F/F_i–bordant.*

Proof. If C is an F/F_i–concordance, then doing surgery along $C \subset S^{2q+1} \times [0,1]$ gives a homology F/F_i–bordism for (M_{L_1}, φ_1) and (M_{L_2}, φ_2). □

The following is immediate from theorem 2.2, lemma 3.5 and propositions 3.6, 3.7. The theorem generalizes results on link concordance by Cha and Ko [CK99].

Theorem 3.8. *Let L_1, L_2 be concordant links. If φ_1, φ_2 are arbitrary F/F_i–structures for L_1, L_2, then there exists a special automorphism ψ of F/F_i such that*
$$\rho(M_{L_1}, \varphi_1)(\alpha) = \rho(M_{L_2}, \psi \circ \varphi_2)(\alpha) \text{ for all } \alpha \in P_k(F/F_i)$$

3.3. **Representations of F/F_2.** We now give an example of a non-trivial (i.e. non-abelian) unitary representation of F/F_2. For $U_1, \dots, U_m \in U(k)$ define $\alpha_{(U_1,\dots,U_m)} : F \to U(k)$ by $\alpha(t_i) := U_i$. We'll find $U_1, \dots, U_m$ such that $\alpha_{(U_1,\dots,U_m)}$ factors through F/F_2.

Let $z_1, \dots, z_k \in S^1$ and $\chi : F \to S^1$ a character such that $\chi(t_i^k) = 1$. Define

$$U_1 := \begin{pmatrix} 0 & \dots & 0 & z_k \\ z_1 & \dots & 0 & 0 \\ 0 & \ddots & & \vdots \\ 0 & \dots & z_{k-1} & 0 \end{pmatrix}, \quad U_i := \begin{pmatrix} \chi(t_i) & 0 & \dots & 0 \\ 0 & \chi(t_1 t_i) & & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t_1^{k-1} t_i) \end{pmatrix}, i = 2, \dots, m$$

Lemma 3.9. *The representation $\alpha = \alpha_{(U_1, \dots, U_m)} : F \to U(k)$ factors through F/F_2.*

Proof. It is clear that we are done once we show that for all $x \in [F, F]$, $\alpha(x) \in \mathbb{C} \cdot \mathrm{id}$. Since

$$[x, vw] = [x, v]v[x, w]v^{-1}$$

we only have to show that $\alpha([x_i, x_j]) \in \mathbb{C} \cdot \mathrm{id}$, but an easy calculation using $\chi(t_i^k) = 1$ shows that

$$\begin{array}{rcll} \alpha([t_1, t_j]) & = & \chi(t_1^{-1}) \cdot \mathrm{id} & \text{if } j \neq 1 \\ \alpha([t_j, t_1]) & = & \chi(t_1) \cdot \mathrm{id} & \text{if } j \neq 1 \\ \alpha([t_i, t_j]) & = & \mathrm{id} & \text{if } i \neq 1 \text{ and } j \neq 1 \end{array}$$

□

Let p a prime, k a power of p, and $z_1, \dots, z_k, \chi$ such that $z_1^{p^N} = \dots = z_k^{p^N} = 1$ and $\chi(v)^{p^N} = \mathrm{id}$ for some N, then $\varphi \in P_k(F/F_2)$. Such a representation turns out to discover non-slice knots in many interesting cases.

This example can easily be generalized to give more complex representations of F/F_2.

3.4. **Sliceness obstructions.**

Theorem 3.10. *Let $L \subset S^{2q+1}$ be a slice link and let $\alpha \in P_k(\pi_1(M_L))$, then $\eta_\alpha(M_L) = 0$.*

Proof. Assume that α factors through a p-group P. Then $P_i = \{e\}$ for some i since any p-group is nilpotent (cf. [J97, p. 169]). In particular α factors through $\pi_1(M_L)/\pi_1(M_L)_i$ which is isomorphic to F/F_i since any slice link has zero $\bar{\mu}$–invariants by proposition 3.6. Henc $\alpha = \beta \circ \varphi$ for some F/F_i–structure φ and some representation β. The statement now follows immediately from proposition 3.6, theorem 3.8 and lemma 3.1 since a slice link is concordant to the unlink. □

Define $PD(k) \subset U(k)$ to be the subgroup generated by permutation matrices and diagonal matrices. For a prime p define $PD_p(k) \subset PD(k)$ to be the subgroup of matrices where all eigenvalues are roots of unity of order a power of p. It is generated by all permutation matrices whose order is a power of p and all diagonal matrices whose entries are roots of unity of order a power of p. Note that a finitely generated subgroup $PD_p(k)$ is in fact a finite group, hence a p-group.

Theorem 3.11. *Let $L \subset S^{2q+1}$ be a slice link with meridians $\mu_1, \dots, \mu_m$. Let p be a prime number and let $U_1, \dots, U_m \in PD_p(K)$. Then there exists a unique representation $\beta : \pi_1(M_L) \to U(k)$ with $\beta(\mu_j) = U_j$. Furthermore $\eta_\beta(M_L) = 0$.*

Proof. Let $\alpha := \alpha(U_1, \dots, U_m) : F \to U(k)$, then $\mathrm{Im}(\alpha)$ is a p-group, hence α factors through F/F_i for some i. It's clear that β is given by $\pi_1(M_L)/\pi_1(M_L)_i \cong F/F_i \to U(k)$. Furthermore $\beta \in P_k(\pi_1(M_L))$, the theorem now follows from theorem 3.10. $\square$

Proposition 3.12. *Let $\alpha \in P_k(F/F_i)$, then there exists a prime p such that α is conjugate to a representation $\tilde{\alpha}$ with $\tilde{\alpha}(t_j) \in PD_p(k)$ for all j.*

Proof. This follows from the fact that if $\alpha : P \to U(k)$ is a representation of a p-group P, then α is induced from a representation of degree 1 (cf. [H67, p. 578ff]). This means that there exists a subgroup $Q \subset P$ and a one-dimensional representation $Q \to U(\mathbb{C})$ such that α is given by the natural P-left action on $\mathbb{C}P \otimes_{\mathbb{C}Q} \mathbb{C}$. Pick representatives $p_1, \dots, p_k$ for P/Q, writing α with respect to this basis we see that α is of the required type. $\square$

Remark. The above proposition together with theorem 3.10 shows that theorem 3.11 is the best possible sliceness obstruction theorem which can be based on theorem 2.2.

3.5. **Algebraic closures of groups and link concordance.** Whereas theorem 3.11 can't be improved on with our means there's still room for improvement for proposition 3.8 because of the extra indeterminacy given by the special automorphism group.

For a group G Levine [L89a], [L89b], [L90] introduced the notion of algebraic closure $\hat{G}$ and residually nilpotent algebraic closure $\bar{G}$ of a group G. The results of section 3 for $G = F/F_i$ also hold for $G = \hat{F}$ and $G = \bar{F}$ (cf. [L94] for details), in particular links with zero $\bar{\mu}$-invariants have a $\bar{F}$-structure and concordant links are also $\bar{F}$-concordant, same for $\hat{F}$. In particular we get link concordance invariants from representations in $P_k(\bar{F})$ and $P_k(\hat{F})$.

Note that p-groups are nilpotent and hence its own algebraic closure ([L90, p. 100]). This shows that representations in $P_k(\pi_1(M_K))$ that factor through an F/F_i–structure for some i correspond to representations that factor through some $\bar{F}$–structure (or $\hat{F}$–structure).

The following theorem is a stronger version of 3.8

Theorem 3.13. *Let L_1, L_2 be concordant links with vanishing $\bar{\mu}$–invariants. If φ_1, φ_2 are arbitrary $\bar{F}$–structures for L_1, L_2, then there exists a special automorphism ψ of $\bar{F}$ such that*

$$\rho(M_{L_1}, \varphi_1)(\alpha) = \rho(M_{L_2}, \psi \circ \varphi_2)(\alpha) \text{ for all } \alpha \in P_k(\bar{F})$$

3.6. **Relation to previous link concordance invariants.** One can easily see that theorem 3.11 contains the sliceness obstructions defined by Smolinsky [S89], [S89b].

We quickly recall a results by Cha and Ko and show how it follows from our results.

Theorem 3.14. [CK99, thm. 7] *Let L be a slice link and p a prime. Let $\varphi : \pi_1(M_L) \to G$ be a homorphism to a finite abelian p-group G. Denote the G-fold cover of M_L by M_G. Let $\alpha_G : H_1(M_G) \to \mathbb{Z}/p \to U(1)$ be a representation that factors through $\mathbb{Z}$, then*

$$\eta(M_G, \alpha_G) = 0$$

Proposition 3.15. *If a link L satisfies the conclusion of theorem 3.10 then it also satisfies the conclusion of theorem 3.14.*

Proof. Let $s = |G|$. By theorem 6.1 there exists a unitary representation $\alpha : \pi_1(M_L) \to U(s)$ such that

$$\eta_{\alpha_G}(M_G) = \eta_\alpha(M_L) - s\eta_{\alpha(G)}(M_L)$$

where $\alpha(G)$ stands for the representation $\pi_1(M_L) \to U(\mathbb{C}[\pi_1(M_L)/\pi_1(M_G)]) = U(\mathbb{C}G)$ given by left multiplication. Furthermore $\alpha \in P_s(\pi_1(M))$ by lemma 6.2 and $\alpha(G) \in P_1(\pi_1(M))$ since G is of prime power order.

If a link L satisfies the conclusion of theorem 3.10, then $\eta_{\alpha(G)}(M_L) = 0$ and $\eta_\alpha(M_L) = 0$. □

In later, unpublished work Cha showed that if L is a slice link, p a prime power, M' a p^a–cover of M_L (not necessarily regular) and $\alpha' : H_1(M') \to U(1)$ a character whose order is a power of p, then $\eta(M', \alpha') = 0$. In this case we can find $M_L = M_0 \subset M_1 \subset \cdots \subset M_k = M'$ such that M_i/M_{i-1} is a regular p–covering. Using lemma 6.2 and theorem 6.1 one can inductively write $\eta(M', \alpha)$ as a sum of eta invariants of M_L with representations factoring through p–groups. This shows that Cha's extended result is contained in theorem 3.10.

4. Eta-invariants and signatures of boundary links

4.1. **Eta-invariants as boundary link concordance invariants.** In this section we denote the free group on m generators once again by F_m. Let $(L, \varphi) \subset S^{2q+1}$ be an F_m–link. If $q > 1$ then $\pi_1(S^{2q+1} \setminus L) \to \pi_1(M_L)$ is an isomorphism. If $q = 1$, then $\varphi(\lambda) = e$ for any longitude, since $[\lambda_i, \mu_i] = 1 \in \pi_1(S^3 \setminus L)$. In particular for any q the map φ factors through $\pi_1(M_L)$.

Proposition 4.1. [L94, p. 102] *Let $(L_1, \varphi_1), (L_2, \varphi_2)$ be F_m-concordant links, then $(M_{L_1}, \varphi_1), (M_{L_2}, \varphi_2)$ are homology F_m-bordant.*

The following theorem is immediate from lemma 1.1, proposition 1.2, theorem 2.2 and the above proposition.

Theorem 4.2. *Let (L_1, φ_1) and (L_2, φ_2) be F_m-concordant F_m-links, then*

$$\rho(M_{L_1}, \varphi_1)(\alpha) = \rho(M_{L_2}, \varphi_2)(\alpha) \text{ for all } \alpha \in P_k(F_m)$$

If L_1, L_2 are boundary concordant boundary links with two components, then

$$\rho(M_{L_1}, \varphi_1)(\alpha) = \rho(M_{L_2}, \varphi_2)(\alpha) \text{ for all } \alpha \in P_k(F_2)$$

for any F_2-structures φ_1 and φ_2.

The following is immediate from theorem 3.10.

Theorem 4.3. *If L is a boundary link, and L is slice (in particular if L is boundary slice), then*

$$\rho(M_L, \varphi)(\alpha) = 0 \text{ for any } \alpha \in P_k(F_m)$$

for any F_m-structure φ.

Corollary 4.4. *If L_1, L_2 are boundary link concordant boundary links and if φ_1, φ_2 are F_m-structures, then there exists a special automorphism $\psi \in CA_m$ such that*

$$\rho(M_{L_1}, \varphi_1)(\alpha) = \rho(M_{L_1}, \psi \circ \varphi_1)(\alpha) \text{ for any } \alpha \in P_k(F_m)$$

Proof. Levine [L94, p. 102] showed that if L_1 is an F_m-link and L_2 a boundary link which is boundary link concordant to L_1, then there exists an F_m-structure on L_2 such that L_1 and L_2 are F_m-concordant. The corollary now follows from lemma 1.1 and theorem 4.2. □

In section 7 we compute the ρ-invariant for an F_m-link. This will involve the computation of the eta-invariant of a circle which necessitates the definition of the following function. Let $z = e^{2\pi i a} \in S^1$ with $a \in [0,1)$, then define

$$\eta(z) := \begin{cases} 0 & \text{if } a = 0 \\ 1 - 2a & \text{if } a > 0 \end{cases}$$

Now we can formulate the following theorem which will be proven in section 7.

Theorem 4.5. *Let $(L \subset S^{2q+1}, \varphi)$ be an F_m-link, $A = (A_{ij})_{i,j=1,\dots,m}$ a Seifert matrix for (L, φ) of size $(g_1, \dots, g_m)$, $\alpha : F_m \to U(k)$ a representation. Let $\epsilon := (-1)^{q+1}, g := \sum_{i=1}^m g_i, T := diag(t_1, \dots, t_1, \dots, t_m, \dots, t_m)$ where each t_i appears $2g_i$ times. Let $\{z_{ij}\}_{j=1,\dots,k}$ be the set of eigenvalues of $\alpha(t_i)$. Then*

$$\begin{aligned} \rho(M_L, \varphi)(\alpha) \;\; = & \;\; \epsilon \sum_{i=1}^m sign(\sqrt{\epsilon}(A_{ii} + \epsilon A_{ii}^t)) \sum_{i=1}^m \sum_{j=1}^k \eta(z_{ij}) + \\ & + \;\; sign(\sqrt{-\epsilon}(A - \epsilon\alpha(T)A^t\alpha(T)^{-1} - A\alpha(T)^{-1} + \epsilon\alpha(T)A^t)) \end{aligned}$$

where we consider A as a $2gk \times 2gk$ matrix, where each entry of $A = (a_{ij})$ is replaced by $a_{ij} \cdot id_k$. This simplifies for $\epsilon = -1$ to the following

$$\rho(M_L, \varphi)(\alpha) = sign(A + \alpha(T)A^t\alpha(T)^{-1} - A\alpha(T)^{-1} - \alpha(T)A^t)$$

Note that if we let $U_i := \alpha(t_i)$, then $A - \epsilon\alpha(T)A^t\alpha(T)^{-1} - A\alpha(T)^{-1} + \epsilon\alpha(T)A^t$ equals

$$\begin{pmatrix} A_{11}(1 - U_1^{-1}) - \epsilon A_{11}^t(1 - U_1) & A_{12}(1 - U_1)(1 - U_2^{-1}) & \dots \\ A_{21}(1 - U_2)(1 - U_1^{-1}) & A_{22}(1 - U_2^{-1}) - \epsilon A_{22}^t(1 - U_2) & \\ \vdots & & \ddots \end{pmatrix}$$

here we use the convention of the theorem again, i.e. we view A_{ij} as a $2g_ik \times 2g_jk$–matrix. Alternatively we could write $A_{11} \otimes (1 - U_1^{-1}) - \epsilon A_{11}^t \otimes (1 - U_1)$ etc..

This result generalizes a computation done by Cha and Ko [CK99] for certain unitary representations. We suggest the following conjecture which would be a generalization of theorem 3.4.

Conjecture 4.6. *Let $q > 1$ and $(L, \varphi) \subset C_{2q+1}(F_m)$. If for all k, $\rho(M_L, \varphi)(\alpha) = 0$ for a dense set of representations $\alpha \in R_k(F_m)$, then (L, φ) represents a torsion element.*

Note that in light of theorem 1.4 this conjecture is purely algebraic. This conjecture seems to be hard to prove, and any attempt would I think require a good understanding of non-commutative algebraic geometry. If this conjecture can be proven to be true then this would give an algorithm for detecting non–torsion elements in $C_n(F_m)$, which is easier to implement than Sheiham's [S02] algorithm. The disadvantage of such an algorithm would be that it can not conclude in finite time that an F_m–link is torsion. An interesting follow up question to a positive answer would be whether there exists a k depending computably on (L, φ), such that it is enough to study the ρ–invariant for dimensions less or equal than k for deciding whether (L, φ) is torsion or not.

4.2. **Signature invariants for boundary link matrices.** Recall that if a boundary link (L, V) is boundary link slice then any Seifert matrix is metabolic. Using this fact and some algebra we can strengthen theorem 4.3.

Let $A = (A_{ij})$ be an ϵ-Seifert matrix and $U_i \in U(k), i = 1, \ldots, m$. We denote by $U := \text{diag}(U_1, \ldots, U_m)$ the block diagonal matrix with blocks $U_i \cdot \text{id}_{h_i}$ and define

$$M(A, U) := \sqrt{-\epsilon}(A - \epsilon U A^t U^{-1} - A U^{-1} + \epsilon U A^t)$$

using the convention of theorem 4.5. Furthermore let $\sigma(A, U) := \text{sign}(M(A, U))$. If A is metabolic then $M(A, U)$ is metabolic as well, if U is such that $\det(M(A, U)) \neq 0$ then $\sigma(A, U) = 0$. The map σ is continuous outside of the set

$$S_k(A) := \{(U_1, \ldots, U_m) \in U(k)^m | \det(M(A, U)) = 0\}$$

It is easy to see that if A_1, A_2 are S-equivalent, then $\sigma(A_1, U) = \sigma(A_2, U)$ and $S(A_1) = S(A_2)$. In particular for a boundary link pair (L, V) we can define $\sigma(L, V, U) := \sigma(A, U)$ using any Seifert matrix and we let $S_k(L, V) := S_k(A)$. This generalizes signature invariants for knots defined by Levine [L69] and Trotter [T73].

We immediately get the following proposition.

Proposition 4.7. *Let (L, V) be a boundary link pair which represents zero in $C_n(B_m)$, then $\sigma(L, V, (U_1, \ldots, U_m)) = 0$ for all $(U_1, \ldots, U_m) \notin S_k(L, V)$.*

Combining this with theorem 4.5 we get a theorem that gives a much stronger boundary sliceness obstruction than theorem 4.3 since the matrices U_i no longer have to lie in $PD_p(k)$ for some prime p.

Theorem 4.8. *Let (L, V) be a boundary link pair which represents zero in $C_n(B_m)$, then*

$$\rho(M_L, \varphi)(\alpha_{(U_1, \ldots, U_m)}) = 0 \textit{ for all } (U_1, \ldots, U_m) \notin S(L, V)$$

Remark. Note that $\sqrt{-\epsilon}(A-\epsilon UA^tU^{-1}-AU^{-1}+\epsilon UA^t)=\sqrt{-\epsilon}(A+\epsilon UA^t)(1-U^{-1})$, using an argument as in [L69] one can show in a purely algebraic way that σ is continuous outside of the set $\{(U_1,\dots,U_m)\in U(k)^m|\det(A+\epsilon UA^t)=0\}$. This agrees with the topological result of Levine's [L94] that ρ is continuous on the set

$$\{\alpha=\alpha_{(U_1,\dots,U_m)}|H_1^\alpha(M_L,\mathbb{C}^k)=0\}$$

since $A+\epsilon TA^t$ represents the homoloy of the universal F_m–cover of M_L. This shows in particular that ρ is zero in a neighborhood of the trivial representation.

Remark. Levine announced a proof that these ρ–invariants are in fact obstructions to a link being slice and not just obstructions to a link being boundary slice.

There are many ways to associate a hermitian matrix to A which is metabolic if A is metabolic. Let $F_{ij}\in M(k_i\times k_j,\mathbb{C}), i,j=1,\dots,m$ such that $F_{ji}=\sqrt{-\epsilon}\bar{F}_{ij}^t$, then we also get a similar proposition for

$$\sigma(A,F_{ij}):=\operatorname{sign}\begin{pmatrix} A_{11}F_{11}+A_{11}^t\bar{F}_{11}^t & A_{12}F_{12} & \cdots \\ A_{21}F_{21} & A_{22}F_{22}+A_{22}^t\bar{F}_{22}^t & \\ \vdots & & \ddots \end{pmatrix}$$

This approach has the advantage that it is much easier to find (random) matrices in $M(k_i\times k_j,\mathbb{C})$ than matrices in $U(k)$.

In the knot case one can easily show that these signature invariants have the same information content as $\sigma(A,U)$. In the case $m>1$ we don't know whether these different signature functions have different information content or not.

5. Examples

Ko [K87] gives an example of a three component boundary link $L\subset S^{4l+3}$ with Seifert manifold V such that $(L_{1,-2},V_{1,-2}):=(L,V)\#-(L,\alpha_{12}V)$ (cf. [K87] for details on the action of CA_3 on Seifert surfaces) has the following Seifert matrix of size $(1,2,2)$

$$A=\begin{pmatrix}
0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&1&0&0&0\\
0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&-1&0\\
0&0&0&0&0&1&0&0&0&0\\
0&1&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&-1&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

Ko showed that $L_{1,-2}$ is not boundary slice and posed the question whether $L_{1,-2}$ is slice or not. By construction we get for $\alpha \in P_1(F_3)$ that

$$\rho(M_{L_{1,-2}}, \varphi_{V_{1,-2}})(\alpha) = \rho(M_L, \varphi_V)(\alpha) - \rho(M_L, \alpha_{12}\varphi_V)(\alpha)\rho(M_L, \varphi_V)(\alpha) - \rho(M_L, \varphi_V)(\alpha \circ \alpha_{12}) = 0$$

since $U(1)$ is abelian. Hence all one–dimensional eta–invariants vanish.

Cha and Ko [CK99] showed that L is in fact not slice. We reprove this using higher dimensional representations. Let

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\ 0 & -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{pmatrix}, \quad U_3 = U_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\ 0 & -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{pmatrix}$$

A computation using theorem 4.5 shows that $\rho(M_L, \varphi)(\alpha_{U_1,U_2}) = -2$, hence L is not slice by theorem 3.11.

On the other hand, let $(L_{1,-1}, V_{1,-1}) := (L, V)\# - (L, V)$, then $L_{1,-1}$ is obviously slice. This proves the following proposition.

Proposition 5.1. *Connected sum is not a well-defined operation on $C(n,m)$ for $m \geq 3$.*

We now give an example of a two component link with vanishing one–dimensional eta–invariant but which is not slice. Consider the following boundary link Seifert matrix of size $(2, 1)$:

$$A = (A_{ij})_{i,j=1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Let $(L, V) = (L_1 \cup L_2, V_1 \cup V_2) \subset S^{4l+3}$ be a boundary link pair with Seifert matrix A. In fact we can find isotopic slice knots L_1, L_2 and corresponding Seifert surfaces with the above property since one can easily see that A_{11} and A_{22} are S-equivalent and metabolic.

Note that $\Delta(L)(t_1, t_2) = \det(AT - A^t)t_1^{-2}t_2^{-1} = -(t_1t_2 + t_1^{-1}t_2^{-1}) - (t_1^{-1}t_2 + t_1t_2^{-1}) + 5$. Let $(\tilde{L}, \tilde{V}) = (L_2, V_2) \cup (L_1, V_1)$, clearly $(\tilde{L}, \tilde{V})$ is a boundary link with Seifert matrix $(\tilde{A}_{ij}) = (A_{ji})$.

Now pick an arc connecting L and $\tilde{L}$ which lies outside of V and $\tilde{V}$. Use this arc to form $L\# - \tilde{L}$. If $q > 1$ then this link is independent of the choice of the arc.

Proposition 5.2. *The boundary link $(L\# - \tilde{L}, V\# - \tilde{V})$ has zero $U(1)$-eta invariants but is not boundary link slice. Furthermore $L\# - \tilde{L}$ is not slice.*

Proof. Let $B = A \oplus -\tilde{A}$ be a Seifert matrix for $(L\# - \tilde{L}, V\# - \tilde{V})$. For $z_1, z_2 \in S^1$ let $Z = \mathrm{diag}(z_1, z_1, z_1, z_1, z_2, z_2)$, then

$$\rho(M_{L\#-\tilde{L}}, \epsilon)_{\alpha_{(z_1,z_2)}} = \mathrm{sign}(B(1 - Z) + B^t(1 - Z^{-1})) = \mathrm{sign}((BZ - B^t)(Z^{-1} - 1))$$

In particular the function $\rho(M_{L\#-\tilde{L}},\epsilon):R_1(\mathbb{Z}^2)=S^1\times S^1\to\mathbb{Z}$ is constant outside of the set $\{(z_1,z_2)\in S^1\times S^1|\det(AZ-A^t)=0\}$. It is obvious that for all $z_1,z_2\in S^1$ we have

$$\det(BZ-B^t)z_1^{-2}z_2^{-1}=(-(z_1z_2+z_1^{-1}z_2^{-1})-(z_1^{-1}z_2+z_1z_2^{-1})+5)^2\geq 1$$

hence the ρ-invariant function is constant. Picking $z_1=-1,z_2=-1$ we can compute that the constant is in fact 0.

Now let

$$U_1=\begin{pmatrix}0&i\\1&0\end{pmatrix},\quad U_2=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$$

A computation using theorem 4.5 shows that $\rho(M_{L\#-\tilde{L}},\varphi)(\alpha_{U_1,U_2})=-2$, hence $L\#-\tilde{L}$ is not slice by theorem 3.11. □

Now consider the following Seifert matrix of size $(1,1)$

$$A=\begin{pmatrix}0&1&0&1\\0&0&2&1\\0&2&1&0\\1&1&1&0\end{pmatrix}$$

Let (L,V) be a boundary link pair with Seifert matrix A. If we let

$$F_{11}=\begin{pmatrix}4&1\\3&0\end{pmatrix},\quad F_{12}=\begin{pmatrix}4&1\\2&1\end{pmatrix},\quad F_{22}=\begin{pmatrix}1&2\\4&1\end{pmatrix}$$

then $\sigma(A,F_{ij})=-2$, which shows that A is not metabolic, i.e. L is not boundary slice. Computer computations indicate that $\rho(M_L,\varphi)$ vanishes on $S_1(L,V)$ and $S_2(L,V)$ but is non-zero on $S_3(L,V)$ which shows again that L is not boundary slice by proposition 4.7.

All the ρ-invariants of theorem 3.11, i.e. all eta invariants corresponding to representations that factor through a p–group that I computed so far with a computer vanish. So it seems like one can not use theorem 3.11 to say that L is not slice.

A new result by Levine (cf. the second remark after theorem 4.8) shows that L is in fact not slice.

6. Relating eta-invariants of finite covers

Let M be an oriented Riemannian manifold of dimension $2l-1$ and $\alpha:\pi_1(M)\to U(k)$ a representation. Denote the universal cover of M by $\tilde{M}$. Then let $V_\alpha:=\tilde{M}\times_{\pi_1(M)}\mathbb{C}^k$, this is a $\mathbb{C}^k$-bundle over M. On the space of differential forms of even degree there's a natural self-adjoint operator B defined by

$$\begin{aligned}\Omega_{2k}(M)&\to\Omega_{2l-2k}(M)\\ \omega&\mapsto i^l(-1)^{k+1}(*d-d*)\omega\end{aligned}$$

This can be naturally extended to give a self-adjoint operator B_α acting on even forms with coefficients in the flat vector bundle defined by α. Consider the spectral function $\eta_\alpha(M,s)$ of this operator defined by

$$\eta_\alpha(M,s) := \sum_{\lambda\neq 0}(\mathrm{sign}(\lambda))|\lambda|^{-s}$$

where λ runs over the eigenvalues of B_α. Atiyah-Patodi-Singer [APS75] showed that for s with $\mathrm{Re}(s)$ big enough, $\eta_\alpha(M,s)$ converges to a holomorphic function. Furthermore this holomorphic function can be extended to 0 and $\eta_\alpha(M,0)$ is finite. Now define the (reduced) eta–invariant of (M,α) to be

$$\eta_\alpha(M) := \eta_\alpha(M,0) - k\eta(M,0)$$

where $\eta(M,s)$ denotes the eta function corresponding to the trivial one–dimensional representation of $\pi_1(M)$. Atiyah-Patodi-Singer [APS75] showed that $\eta_\alpha(M)$ is independent of the Riemannian metric on M.

Let M be a manifold of dimension $2l-1$ and M' a finite cover, not necessarily regular. Let $\alpha' : \pi_1(M') \to U(k)$ be a representation. The goal is to express $\eta_{\alpha'}(M')$ in terms of eta-invariants of M.

Consider $\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M')} \mathbb{C}^k$ where we view $\mathbb{C}^k$ as a $\mathbb{C}\pi_1(M')$-module via α'. We give $\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M')} \mathbb{C}^k$ the metric induced by

$$((p_1 \otimes v_1),(p_2 \otimes v_2)) \to \sum_{g\in\pi_1(M')} \delta_{(p_1g,p_2)}\overline{(\alpha'(g)^{-1}v_1)}^t v_2$$

wgere $p_i \in \pi_1(M), v_i \in \mathbb{C}^k$. It's easy to see that this is well-defined. Let $s := [\pi_1(M) : \pi_1(M')]$, then clearly $\dim(\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M')} \mathbb{C}^k) = ks$.

Define

$$\begin{array}{rcl} \alpha : \pi_1(M) & \to & \mathrm{Aut}(\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M')} \mathbb{C}^k) \\ a & \mapsto & (p \otimes v \mapsto ap \otimes v) \end{array}$$

This action is obviously isometric, i.e. unitary.

Denote by $\alpha(M,M')$ the representation $\pi_1(M) \to U(\mathbb{C}\pi_1(M) \otimes_{\pi_1(M')} \mathbb{C})$ given by left multiplication where we consider $\mathbb{C}$ as the trivial $\pi_1(M')$–module.

Theorem 6.1.

$$\eta_{\alpha'}(M') = \eta_\alpha(M) - k\eta_{\alpha(M,M')}(M)$$

Proof. Give M some Riemannian structure and M' the induced structure. We have to show that

$$\eta_{\alpha'}(M',0) - k\eta(M',0) = (\eta_\alpha(M,0) - ks\eta(M,0)) - k(\eta_{\alpha(M,M')}(M,0) - s\eta(M,0))$$

We'll in fact show that

$$\begin{array}{rcl} \eta_{\alpha'}(M',0) & = & \eta_\alpha(M,0) \\ \eta(M',0) & = & \eta_{\alpha(M,M')}(M,0) \end{array}$$

Recall that

$$\begin{array}{rcl} V_{\alpha'} & = & \tilde{M} \times_{\pi_1(M')} \mathbb{C}^k \text{ and} \\ V_\alpha & = & \tilde{M} \times_{\pi_1(M)} (\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M')} \mathbb{C}^k) \end{array}$$

Let $p \in M, U \subset M$ a (small) neighborhood and $p_1, \dots, p_s, U_1, \dots, U_s$ the different lifts. Then the map

$$\begin{array}{rcl} \oplus_{i=1}^s \pi_i : V_\alpha|_U & \to & \oplus_{i=1}^s V_{\alpha'}|_{U_i} \\ \sum (q, g_i h_i \otimes v_i) & \mapsto & \sum (q g_i, h_i v_i) \end{array}$$

is an isomorphism with inverse map given by

$$\sum (q_i, 1 \otimes v_i) \leftarrow\!\shortmid \sum (q_i, v_i)$$

where g_i such $\pi(q g_i) \in U_i$ and $h_i \in \pi_1(M')$. Note that

$$\Omega^{2i}(M', V_\alpha)|_{\cup U_i} = \oplus_{i=1}^s \Omega^{2i}(M')|_{U_i} \otimes_{C^\infty(U_i)} \Gamma(V_{\alpha'}|_{U_i})$$

This is isomorphic to

$$\Omega^{2i}(M)|_U \otimes_{C^\infty(U)} \oplus_{i=1}^s \Gamma(V_{\alpha'}|_{U_i}) \cong \Omega^{2i}(M)|_U \otimes_{C^\infty(U)} \Gamma(V_\alpha|_U)$$

which is just $\Omega^{2i}(M, V_\alpha)|_U$. It's clear that these isomorphisms can be patched together and give an isomorphism $\Omega^{2i}(M, V_\alpha) \cong \Omega^{2i}(M', V_{\alpha'})$ which commutes with $*$ and d since these operators are defined locally. $\eta_\alpha(M, s) = \eta_{\alpha'}(M', s)$, hence $\eta_\alpha(M, 0) = \eta_{\alpha'}(M', 0)$

Exactly the same way using the trivial representation for α' one shows that $\eta(M', 0) = \eta_{\alpha(M,M')}(M, 0)$. □

In the application we'll have the case that $\pi_1(M') \subset \pi_1(M)$ is normal. We'll now restrict ourselves to this case. Write $G := \pi_1(M')/\pi_1(M)$ and write M_G, α_G for M' and α'. We'll give an explicit matrix representation for α_G and show that if $\alpha_G \in P_k(\pi_1(M_G))$ then $\alpha \in P_{ks}(\pi_1(M))$.

Let $g_1, \dots, g_s$ be the elements of G and pick a splitting $\psi : G \to \pi_1(M)$ which is of course in general not a homomorphism, but we can arrange ψ such that $\psi(g^{-1}) = \psi(g)^{-1}$ and $\psi(e) = e$. Let $e_1, \dots, e_k$ denote the canonical basis of $\mathbb{C}^k$. Then $g_i \otimes e_j$ is a basis for $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^k$ and $\psi(g_i) \otimes e_j$ is a basis for $\mathbb{C}\pi_1(M) \otimes_{\mathbb{C}\pi_1(M_G)} \mathbb{C}^k$. We'll write α with respect to the basis $\psi(g_i) \otimes e_j$. Note that

$$a\psi(g_i) \otimes v = \psi(\varphi(a)g_i)\psi(\varphi(a)g_i)^{-1} a\psi(g_i) \otimes v = \psi(\varphi(a)g_i) \otimes \beta(\psi(\varphi(a)g_i)^{-1} a\psi(g_i))v$$

since $\varphi(\psi(\varphi(a)g_i)^{-1} a\psi(g_i)) = 1$. Therefore $\alpha(a)$ is given by

$$P_{\varphi(a)} \begin{pmatrix} \alpha_G(\psi(g_1\varphi(a))^{-1} a\psi(g_1)) & 0 & \dots & 0 \\ 0 & \alpha_G(\psi(g_2\varphi(a))^{-1} a\psi(g_2)) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_G(\psi(g_s\varphi(a))^{-1} a\psi(g_s)) \end{pmatrix}$$

where $P_{\varphi(a)} : \mathbb{C}G \to \mathbb{C}G$ denotes the matrix corresponding to left multiplication by $\varphi(a)$, i.e. $P_{\varphi(a)}(\psi(g_i) \otimes e_j) = \psi(\varphi(a)g_i) \otimes e_j$.

Assume that α_G factors through a homomorphism $\tilde{\alpha}_G$ to a group H_G, then α factors through

$$H := \{(\varphi(a), (h \mapsto (\tilde{\alpha}_G(\psi(h\varphi(a))^{-1} a \psi(h)))_{h\in G}) | a \in \pi_1(M)\} \subset G \ltimes H_G^G$$

where $H_G^G = \mathrm{Maps}(G, H_G)$ and G acts on H_G^G by precomposition by left multiplication. If H_G is a p–group and G is a p–group with respect to the same prime, then H is also a p–group which proves the following lemma.

Lemma 6.2. *Let p a prime number. If G is a p-group and α_G factors through a p-group then α also factors through a p-group, i.e. $\alpha \in P_{ks}(\pi_1(M))$.*

7. Computation of eta-invariants for boundary links

Let (L, φ) be a boundary link pair and $V = V_1 \cup \cdots \cup V_m$ a corresponding Seifert surface. Let $\alpha \in R_k(F_m)$, then define $\theta := \alpha \circ \varphi : \pi_1(M_L) \to F_m \to U(k)$. In this section we'll compute $\rho(M_L, \varphi)(\alpha) = \eta_\theta(M_L)$ using theorem 2.1.

First we add handles $D_i^{2q} \times D^2$ along the L_i to D^{2q+2} and denote this manifold by N_L, then $M_L = \partial(N_L)$. Note that φ does not extend over N_L since in fact $\pi_1(L) = 1$. We push the surfaces V_i into D^{2q+2}, more explicitely, we can find a map $\iota : V \times I \to D^{2q+2}$, $I = [0,1]$, such that $\iota | V \times 0$ is the embedding of V into S^{2q+1}, $\iota | L_i \times I$ is constant on the intervals and such that $\iota|_{int(V)\times I}$ is an embedding. Now let $\Sigma_i := \iota(V \times 1) \cup D_i \times 0 \subset N_L$, $\Sigma := \cup_{i=1}^m \Sigma_i$, and $N := N_L \setminus N(\Sigma)$, then $\partial(N) = M_L \cup -\Sigma \times S^1$.

We can find embeddings $g_i : D^{2q} \times I \hookrightarrow D_i^{2q} \times D^2$ such that $g_i | D^{2q} \times 0$ is just the embedding in $D_i^{2q} \times 0 \subset D_i^{2q} \times D^2$ and such that $g_i | D^{2q} \times 1 \subset M_L$ and $g_i | \partial(D^{2q}) \times I \subset V_i$. Now let $T_i := \iota(V \times I) \cup H_i$ and $T := \cup_{i=1}^m T_i$. Then the Pontrjagin construction for $T \subset N$ gives a map $\pi_1(N) \to F_m$ which extends $\varphi : \pi_1(M_L) \to F_m$. We denote the map $\pi_1(N) \to F_m \to U(k)$ by θ as well. Note that T_i inherits an orientation from $\mathrm{int}(V_i) \times I \subset T_i$. By theorem 2.1

$$\eta_\theta(M_L) - \sum_{i=1}^m \eta_{\tilde{\theta}_i}(\Sigma_i \times S^1) = \mathrm{sign}_\theta(N) - k \cdot \mathrm{sign}(N)$$

where $\tilde{\theta}_i = \theta \circ i_* : \pi_1(\Sigma_i \times S^1) \to \pi_1(N) \to F_m \to U(k)$.

7.1. Computation of $\eta_{\tilde{\theta}_i}(\Sigma_i \times S^1)$.

Note that S^1 inherits an orientation from the orientations of Σ_i and $\Sigma_i \times S^1$. Denote by m_i the (oriented) generator of $\pi_1(S^1)$, then

$$\tilde{\theta}_i : \pi_1(\Sigma_i) \times \pi_1(S^1) \cong \pi_1(\Sigma_i \times S^1) \to U(k)$$

is given by sending (g, m_i^e) to $\alpha(t_i)^e$. We need the following proposition.

Proposition 7.1. [N79, thm. 1.2]

(1) *Let $\alpha_N : \pi_1(N^{2r}) \to U(k_N)$ and $\alpha_X : \pi_1(X^{2s-1}) \to U(k_X)$ be representations, then*

$$\eta_{\alpha_N \otimes \alpha_X}(N^{2r} \times X^{2s-1}) = (-1)^{rs} sign_{\alpha_N}(N) \eta_{\alpha_X}(X)$$

(2) *Let* $\alpha : \pi_1(S^1) = \mathbb{Z} \to U(1)$ *be a representation. If* $\alpha(1) = e^{2\pi i a}, a \in [0,1)$, *then*

$$\eta_\alpha(S^1) = \eta(\alpha(1)) := \begin{cases} 0 & \text{if } a = 0 \\ 1-2a & \text{if } a \in (0,1) \end{cases}$$

Therefore

$$\eta_{\tilde{\theta}_i}(\Sigma_i \times S^1) = \epsilon \text{sign}(\Sigma_i) \sum_{i=1}^{m} \sum_{i=1}^{k} \eta(c_{ij})$$

where $\{c_{ij}\}_{j=1,\dots,m}$ denotes the set of eigenvalues of $\alpha(t_i)$ and $\epsilon := (-1)^q$. We can express $\text{sign}(\Sigma_i)$ in terms of the Seifert matrix as follows:

$$\text{sign}(\Sigma_i) = \text{sign}(V_i) = \text{sign}(\sqrt{\epsilon}(A_{ii} + \epsilon A_{ii}^t))$$

In the case $\epsilon = -1$ one can easily show that $A_{ii} - A_{ii}^t$ is congruent to $\begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}$, hence the signature is zero.

7.2. **Computation of $\mathbf{sign}_\theta(N)$.**

7.2.1. *Computation of* $H^\theta_{q+1}(N, \mathbb{C}^k)$. Denote by $\tilde{N}$ the F_m-cover of N induced by φ, note that $C_*(\tilde{N})$ has a right F_m-module structure. Recall that the twisted homology $H_i^\alpha(N, \mathbb{C}^k)$ is defined as $H_i(C_*(\tilde{N}) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k)$, where $\mathbb{C}^k$ is a left F_m-module via α. Fix an orientation preserving embedding $f : (T, \partial(T)) \times [-1,1] \to (N, \partial(N))$, such that $f(T \times 0)$ is the usual embedding of $T \subset N$. Let $X := N \setminus f(T \times (-1,1))$, then X is homoemorphic to N cut along T. We can embed T in X via the embeddings $f_+(c) := f(c,1)$ and $f_-(c) := f(c,-1)$.Then $\tilde{N} \cong X \times F_m/\sim$, where $f_-(c_i) \times z t_i \sim f_+(c_i) \times z$ for $c_i \in T_i, z \in F_m$. From this decomposition we get the following short exact sequence (where $c_i \in C_*(T_i)$)

$$\begin{array}{ccccc} 0 \to C_*(T \times F_m) & \to & C_*(X \times F_m) & \to & C_*(\tilde{N}) \to 0 \\ (c_i, z) & \mapsto & (f_-(c_i), z t_i) - (f_+(c_i), z) & & \\ & & (c, z) & \mapsto & (c, z) \end{array}$$

We tensor with $\mathbb{C}^k$ over $\mathbb{Z}F_m$, the tensored sequence is still exact since $C_*(\tilde{N})$ is a free $\mathbb{Z}F_m$-module. Taking the long exact homology sequence we get

$$\dots \to H_i^\theta(T, \mathbb{C}^k) \to H_i^\theta(X, \mathbb{C}^k) \to H_i^\theta(N, \mathbb{C}^k) \to H_{i-1}^\theta(T, \mathbb{C}^k) \to \dots$$

where

$$\begin{array}{rcl} H_i^\theta(T, \mathbb{C}^k) & = & H_i(C_*(T \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k) = H_i(C_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^k) = H_i(T, \mathbb{C}^k) \\ H_i^\theta(X, \mathbb{C}^k) & = & H_i(C_*(X \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k) = H_i(C_*(X) \otimes_{\mathbb{Z}} \mathbb{C}^k) = H_i(X, \mathbb{C}^k) \end{array}$$

We have to compute $H_*(X)$. Write $X = X_1 \cup X_2$ where $X_1 := X \cap D^{2q+2}$ and $X_2 := X \cap (\cup_{i=1}^m D_i^{2q} \times D^2)$. X_1 is homotopy equivalent to a point since $X_1 \cong D^{2q+2} \setminus f(T \times (-1,1))$, which is just a deformation retract of D^{2q+2}. Furthermore

$H_*(X_2) = H_*(\cup_{i=1}^m (D_i^{2q} \times D^2 \setminus H_i)) = H_*(m \text{ points})$, so from the Mayer-Vietoris sequence we get for $i \geq 2$

$$H_i(X) \cong H_{i-1}(X_1 \cap X_2) = H_{i-1}(L \times (D^2 \setminus I)) = H_{i-1}(L)$$

furthermore

$$0 \to H_1(X) \to H_0(X_1 \cap X_2) \to H_0(m \text{ points}) \oplus H_0(X_2) \to H_0(X) \to 0$$

so $H_i(X) = 0$ for all $i = 1, \ldots, 2q-1$, $H_0(X) = \mathbb{Z}$ and $H_{2q}(X) = \mathbb{Z}^m$.

Proposition 7.2. *If $q > 1$ or $(\alpha(t_i) - id)$ is invertible for all i, then*

$$H_{q+1}^\theta(N, \mathbb{C}^k) \cong H_q(T \times I, \mathbb{C}^k) \cong H_q(\Sigma, \mathbb{C}^k) \cong H_q(V, \mathbb{C}^k)$$

Proof. The last isomorphism follows since $\Sigma^{2q} = V^{2q} \cup D^{2q}$, the second isomorphism is clear, so it only remains to prove the first isomorphism. For $q \geq 2$ this follows immediately from the long exact sequence. In the case $q = 1$ we get

$$\cdots \to H_2(T, \mathbb{C}^k) \to H_2(X, \mathbb{C}^k) \to H_2^\theta(N, \mathbb{C}^k) \to H_1(T, \mathbb{C}^k) \to H_1(X, \mathbb{C}^k) = 0$$

but the map $H_2(T, \mathbb{C}^k) \to H_2(X, \mathbb{C}^k)$ is induced by the map

$$\begin{array}{rcl} C_2(T \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k & \to & C_2(X \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k \\ (c_i, z) \otimes v & \mapsto & (f_-(c_i), zt_i) \otimes v - (f_+(c_i), z) \otimes v) \end{array}$$

Consider the maps

$$\begin{array}{rcccccccl} f_+, f_- : & \mathbb{Z} = H_2(\Sigma) & \cong & H_2(T) & \to & H_2(X) & \xrightarrow{\cong} & H_1(X_1 \cap H_2) = \mathbb{Z} \\ & [\Sigma] & \to & [\Sigma] & \to & [f_\pm(\Sigma)] & \to & [f_\pm(\Sigma) \cap (X_1 \cap X_2)] \end{array}$$

But $[f_\pm(\Sigma) \cap (X_1 \cap X_2)] = K$, i.e. $f_+ = f_-$. Therefore

$$\begin{array}{rcl} C_2(T \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k & \to & C_2(X \times F_m) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k \\ (c_i, z) \otimes v & \mapsto & (f_-(c_i), z) \otimes (t_i v - v) \end{array}$$

If $(\alpha(t_i) - \mathrm{id})$ is invertible for all i, then $H_{q+1}^\theta(N, \mathbb{C}^k) \cong H_q(V, \mathbb{C}^k)$. $\square$

In the following we will assume that the assumptions of the proposition hold.

We can give a more explicit definition of the isomorphism $H_q(V, \mathbb{C}^k) \to H_{q+1}^\theta(N, \mathbb{C}^k)$. Denote by * the pushing of V into $T = \iota(V \times I)$. Let $\{l_{i1}, \ldots, l_{ih_i}\}$ be bases of the torsion free part of $H_q(V_i)$, fix representatives of l_{ij} in $C_q(V)$ which are in general position, we'll denote them by l_{ij} as well. For $l \in \{l_{ij}, l_{ij}^*\}$ denote by $c^+(l)$ resp. $c^-(l)$ a chain in $X_1 \subset D^{2q+2}$ with $\partial(c^+(l)) = f_+(l)$ respectively $\partial(c^-(l)) = f_-(l)$, we can assume that the chains are in general position to each other. Consider X_1 as lying in $\tilde{N}$ via $X_1 \to X_1 \times e \to \tilde{N}$. Then the map

$$\begin{array}{rcl} \psi : C_i(V) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k & \to & C_{i+1}(N) \otimes_{\mathbb{Z}F_m} \mathbb{C}^k \\ l \otimes v & \mapsto & c^+(l_{ki}) \otimes v - c^-(l_{ki}) t_k \otimes v \end{array}$$

induces the above isomorphism $H_q(V, \mathbb{C}^k) \cong H_{q+1}^\theta(N, \mathbb{C}^k)$.

7.2.2. *The intersection pairing on $H^\theta_{q+1}(N,\mathbb{C}^k)$.* Consider the equivariant intersection pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle : C_{q+1}(\tilde{N})\times C_{q+1}(\tilde{N}) & \to & \mathbb{Z}F_m \\ (c,\tilde{c}) & \mapsto & \sum_{g\in F_m}((c\cdot g)\cdot \tilde{c})g^{-1} \end{array}$$

where $(c\cdot g)\cdot\tilde{c}$ is the ordinary intersection number, which is 0 for almost all g. Note that $\langle cg,\tilde{c}\rangle = \langle c,\tilde{c}\rangle g$ and $\langle c,\tilde{c}g\rangle = g^{-1}\langle c,\tilde{c}\rangle$. Denote by A the Seifert matrix of (L,φ) with respect to the basis $\{l_{i1},\dots,l_{ih_i}\}$.

Lemma 7.3. *We get the following matrix of $\langle\ ,\ \rangle$ with respect to the elements $\psi(l_{ij})$*

$$\begin{array}{rl} & \begin{pmatrix} A_{11}(1-t_1^{-1})-\epsilon A_{11}^t(1-t_1) & A_{12}(1-t_1)(1-t_2^{-1}) & \dots \\ A_{21}(1-t_2)(1-t_1^{-1}) & A_{22}(1-t_2^{-1})-\epsilon A_{22}^t(1-t_2) & \\ \vdots & & \ddots \end{pmatrix} = \\ = & A-\epsilon TA^tT^{-1}-AT^{-1}+\epsilon TA^t = \\ = & (A+\epsilon TA^t)(1-T^{-1}) \end{array}$$

Note that a similar computation has been done by Ko (cf. [K89]) for the intersection form of the (abelian) $\mathbb{Z}^m$-cover of N.

Proof. Denote by $_+$ resp. $_-$ pushing into the positive resp. negative direction in $\mathrm{int}(V)\times[-1,1]\subset S^{2q+1}$ and in $\mathrm{int}(V)\times I\subset T\times I$. The map $\psi : H_q(V,\mathbb{C}^k)\to H^\theta_{q+1}(N,\mathbb{C}^k)$ is induced by $\psi(l_{ki}) = c^+(l_{ki})-c^-(l_{ki})t_k$. We can deform $\psi(l_{ki})$ into $d(\psi(l_{ki})) = c^+(l^*_{ki})-c^-(l^*_{ki})t_k$. Note that

(1) Right multiplication by t_k is an isometry.
(2) $\mathrm{lk}(l,\tilde{l}) = -\epsilon\mathrm{lk}(\tilde{l},l)$.
(3) Denote by $c^+(l_{ki}), c^+(l^*_{lj})\subset X\subset S^{2n+1}\setminus N(L)$ submanifolds representing the corresponding chains. We can add a cylinder in $T\times(-1,1)$ to $c^+(l^*_{lj})$ to get a submanifold c with $\partial(c) = l_{lj}$. Then $\mathrm{lk}(l_{ki+},l_{ij}) = \mathrm{lk}(f_+(l_{ki}),l_{lj}) = c^+(l_{ki})\cdot c = c^+(l_{ki})\cdot c^+(l^*_{lj})$ by the definition of the linking pairing (cf. [R90]).
(4) $c^-(l_{ki})\cdot c^-(l^*_{lj}) = \mathrm{lk}(l_{ki-},l_{lj})$ as above.
(5) $c^\pm(l)z\cdot c^\pm(\tilde{l})\tilde{z} = 0$ if $z\neq\tilde{z}$ and if $l,\tilde{l}$ don't intersect, since the chains don't intersect.
(6) $c^+(l)\cdot c^-(\tilde{l}) = \mathrm{lk}(l_+,\tilde{l}_-)$, $c^-(l)\cdot c^+(\tilde{l}) = \mathrm{lk}(l_-,\tilde{l}_+)$ since the embedding $X\subset S^{2n+1}\setminus N(L)$ doesn't change the intersection numbers, and $f_\pm(l) = l_\pm$ and $f_\pm(\tilde{l}) = \tilde{l}_\pm$.

Using this we compute

$$\begin{array}{rcl} \psi(l_{ki})\cdot\psi(l_{lj}) & = & \psi(l_{ki})\cdot d(\psi(l_{lj})) = (c^+(l_{ki})-c^-(l_{ki})t_k)\cdot(c^+(l^*_{lj})-c^-(l^*_{lj})t_l) = \\ & = & c^+(l_{ki})\cdot c^+(l^*_{lj}) + (-c^-(l_{ki})t_k)\cdot(-c^-(l^*_{lj})t_l) = \\ & = & \mathrm{lk}(l_{ki+},l_{lj})+\mathrm{lk}(l_{ki-},l_{lj})\delta_{kl} \end{array}$$

and for $z\neq 1$ we compute

$$\psi(l_{ki})z\cdot\varphi(l_{lj}) = (c^+(l_{ki})z-c^-(l_{ki})zt_k)\cdot(c^+(l_{lj})-c^-(l_{lj})t_l)$$

this is zero except for the following cases:

$$\begin{array}{rclcl} z & = & t_l & \Rightarrow & \psi(l_{ki})z\cdot\psi(l_{lj}) = c^+(l_{ki})t_l\cdot(-c^-(l_{lj})t_l) = -\mathrm{lk}(l_{ki+},l_{lj}) \\ z & = & t_k^{-1} & \Rightarrow & \psi(l_{ki})z\cdot\psi(l_{lj}) = -c^-(l_{ki})t_kt_k^{-1}\cdot c^+(l_{lj}) = -\mathrm{lk}(l_{ki-},l_{lj}) \\ z & = & t_l^{-1}t_k & \Rightarrow & \psi(l_{ki})z\cdot\psi(l_{lj}) = -c^-(l_{ki})t_lt_l^{-1}t_k\cdot(-c^-(l_{lj})t_l) = \mathrm{lk}(l_{ki},l_{lj}) \\ & & & & \text{since } z\neq 1 \text{ implies } k\neq l \end{array}$$

The lemma now follows immediately from the definition of the Seifert matrix A. □

Recall that the twisted intersection pairing is defined as follows

$$\begin{array}{rcl} (\,,\,): C_{q+1}(\tilde{N})\otimes_{\mathbb{Z}F_m}\mathbb{C}^k \times C_{q+1}(\tilde{N})\otimes_{\mathbb{Z}F_m}\mathbb{C}^k & \to & \mathbb{C} \\ (c\otimes v,\tilde{c}\otimes\tilde{v}) & \mapsto & \bar{\tilde{v}}^t\alpha(\langle c,\tilde{c}\rangle)v \end{array}$$

We now proved the following proposition.

Proposition 7.4. *If $(\alpha(t_i)-id)$ is invertible for all i, then the intersection pairing on $H^\alpha_{q+1}(N,\mathbb{C}^k)$ with respect to the basis $l_{ij}\otimes e_k\in H^\theta_{q+1}(N,\mathbb{C}^k)$ is represented by the matrix*

$$\sqrt{-\epsilon}(A-\epsilon\alpha(T)A^t\alpha(T)^{-1}-A\alpha(T)^{-1}+\epsilon\alpha(T)A^t$$

In particular

$$sign_\theta(N) = sign(\sqrt{-\epsilon}(A-\epsilon\alpha(T)A^t\alpha(T)^{-1}-A\alpha(T)^{-1}+\epsilon\alpha(T)A^t)$$

7.3. **Proof of theorem 4.5.** Recall that we have to show the following.

Claim. Let $(L\subset S^{2q+1},\varphi)$ be an F_m-link, $A=(A_{ij})_{i,j=1,\dots,m}$ a Seifert matrix for (L,φ), $\alpha: F_m\to U(k)$ a representation. Let $\epsilon:=(-1)^{q+1}$, then

$$\begin{array}{rcl} \rho(M_L,\varphi)(\alpha) & = & \epsilon\sum_{i=1}^m \mathrm{sign}(\sqrt{\epsilon}(A_{ii}+\epsilon A^t_{ii}))\sum_{i=1}^m\sum_{j=1}^k\eta(z_{ij})+ \\ & + & \mathrm{sign}(\sqrt{-\epsilon}(A-\epsilon\alpha(T)A^t\alpha(T)^{-1}-A\alpha(T)^{-1}+\epsilon\alpha(T)A^t)) \end{array}$$

Proof. The statement under the assumption that either $q>1$ or $(\alpha(t_i)-\mathrm{id})$ is invertible for all i follows immediately from the calculations above and the observation that the untwisted signature is 0.

In the general case we have

$$H^\theta_{q+1}(N,\mathbb{C}^k)\cong H_q(V,\mathbb{C}^k)\oplus\mathrm{Im}(H_2(X,\mathbb{C}^k)\to H^\theta_2(N,\mathbb{C}^k))$$

Let $(c,z)\in\mathrm{Im}(H_2(X,\mathbb{C}^k)\to H^\theta_2(N,\mathbb{C}^k))$ for $i=1,2$ and $(d,w)\in\mathrm{Im}(H_q(V,\mathbb{C}^k)\to H^\theta_{q+1}(N,\mathbb{C}^k))$. Then $cg\cdot d=0$ since c can be represented by an element which is supported on $\partial(N)$ whereas d can be represented by an element which is supported on $N\setminus\partial(N)$.

Since $H_2(X)$ is generated by $[f_-(\Sigma_i)]$ it remains to show that $f_-(\Sigma_i)g$ and $f_-(\Sigma_j)^*$ are disjoint for any $g\in F_m, i,j\in\{1,\dots,m\}$. That's obvious for $i\neq j$ and for $g\neq e$. Recall that $\Sigma_i\cap(X_1\cap X_2)=K$. Pick a longitude K' for K. Pick a Seifert surface V' for K' and close it by a disk D' in the 2-handle over K. Then $[V'\cup D']$ represents $[\Sigma]$ and we can assume that Σ_i and $V'\cup D'$ are in general position. But

$$\Sigma\cdot(V'\cup D') = (V\cup D^2)\cdot(V'\cup D') = V\cdot V'+D^2\cdot D' = 0$$

since $V \cdot V' = \mathrm{lk}(K, K') = 0$ and D, D' can be chosen to be disjoint.

□

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